EVALUATING STUDENTS’ ANSWERSCRIPTS WITH FUZZY ARITHMETIC

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Abstract

In this paper, we discuss the analysis of students’ evaluation, especially in the case of the students’ answerscripts under the evaluation grade of linguistic data. We transfer mostly linguistic data, subjective message into triangular fuzzy numbers, and use the function principle instead of the extension principle to calculate the students’ score. In addition to, we use the degree of similarity between two fuzzy numbers with the utility value of fuzzy number to transfer the students’ score into letter-grade score.

Key Words: linguistic data, utility value, function principle, letter-grade score.

1. Introduction

In general, human transfers the linguistic data such as good, very good, hot, cool etc., into certain data. It is unsensible and unreasonable. In 1965, Zadeh proposed the Fuzzy sets theory, and in 1978, Dubois and Prade defined the fuzzy number for transferring these linguistic data. Recently, many paper are applied fuzzy sets theory on education for evaluating the teachers and the students.

As in [1], Biswas presented a method for evaluating students’ answerscripts. The method, be named Fuzzy Evaluation Method (FEM), is a computer based fuzzy approach. It use extension principle to calculate the students’ score, and define the degree of similarity between two fuzzy sets and mid-point grade to transfer the students’ score into letter grade.

In this paper, we transfer the evaluation grade of linguistic data into triangular fuzzy number [12]. We also use the function principle [3] to calculate the students’ score. The principle does not change the type of membership function and will reduce the trouble and tediousness of operations. The applications of fuzzy number will then be more acceptable. Last, we define the degree of similarity between two fuzzy numbers with the utility value [4] of fuzzy number to transfer the students’ fuzzy total score into letter-grade score.

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2. Fuzzy operations and utility value

Fuzzy numbers are characterized by the normal and convex membership functions which can be viewed as possibility distributions. A typical fuzzy number is a triangular fuzzy number, be used transferring the linguistic data. Let us describe it as fellows.

**Definition 2.1.** Let \( \tilde{A} \) be a triangular fuzzy number with membership function \( \tilde{\mu}_A(x) \), that be shown Figure 1, and be denoted as \( \tilde{A} = (c, a, d) \), and must satisfy some property as fellows:

1. \( \tilde{\mu}_A(x) \) is continuous function;
2. \( \tilde{\mu}_A(x) = 0, \forall x \in (-\infty, c) \);
3. \( \tilde{\mu}_A(x) \) is a strictly increasing on \([c, a]\);
4. \( \tilde{\mu}_A(x) = 1, x = a \);
5. \( \tilde{\mu}_A(x) \) is a strictly decreasing on \([a, d]\);
6. \( \tilde{\mu}_A(x) = 0, \forall x \in (d, \infty) \);

where \( c, a, b, \) and \( d \) are real numbers, with \( c \leq a \leq d \).

![Figure 1 Membership function of \( \tilde{A} \)](image)

In order to transferring the evaluation grade of linguistic data into triangular fuzzy number, we define the linguistic universal and quantitative fuzzy number as fellows.

**Definition 2.2.** Let \( U \) be a universal of linguistic data, \( U = \{ \text{very very good, superior, very good, good, satisfactory to good, satisfactory, bad, very bad, very very bad} \} \), then \( U \) is a Linguistic Universal. Furthermore, we transfer these linguistic data into triangular fuzzy number as fellow:
very very good = (92, 96, 100)  satisfactory = (58, 65, 72)
superior = (87, 90, 93)  bad = (40, 50, 60)
very good = (82, 85, 88)  very bad = (32, 38, 44)
good = (76, 80, 84)  very very bad = (0, 17, 34)
satisfactory to good = (70, 74, 78)

In addition to, we use the fuzzy arithmetics based on function principle to calculate the students’ evaluation score. We describe the addition and the multiplication operations as fellows:

\[ \tilde{M} \oplus \tilde{N} = (c_1 + c_2, a_1 + a_2, d_1 + d_2) \]
\[ \tilde{M} \otimes \tilde{N} = (c_1 \times c_2, a_1 \times a_2, d_1 \times d_2) \]

where \( \tilde{M} = (c_1, a_1, d_1) \) and \( \tilde{N} = (c_2, a_2, d_2) \), and \( c_1, a_1, d_1, c_2, a_2, d_2 \) are positive real numbers.

**Example.** Let \( \tilde{M} = (3, 4, 5) \) and \( \tilde{N} = (1, 2, 3) \), then the addition, and the multiplication operations of \( \tilde{M} \) and \( \tilde{N} \) are calculating as fellows, and the result be shown in Figure 2.

\[ \tilde{M} \oplus \tilde{N} = (3+1, 4+2, 5+3) = (4, 6, 8) \]
\[ \tilde{M} \otimes \tilde{N} = (3\times1, 4\times2, 5\times3) = (3, 8, 15) \]

![Figure 2](image.png)

In the fellowing, we descript the definitions of the utility value of fuzzy number based on the Maximizing set and Minimizing set \([4]\) of fuzzy numbers \( \tilde{A}_i, i = 1, 2, \ldots, n \). These definition will be applied on calculating the degree of similarity between two fuzzy numbers.
**Definition 2.3.** Maximizing set $M$ and Minimizing set $G$ are two fuzzy sets, be showed in Figure 3, and the membership function $\mu_M$ and $\mu_G$ are fellowing:

\[
\mu_M(x) = \begin{cases} 
\frac{x - x_{\text{min}}}{x_{\text{max}} - x_{\text{min}}} , & x_{\text{min}} \leq x \leq x_{\text{max}} \\
0 , & \text{otherwise} 
\end{cases}
\]

\[
\mu_G(x) = \begin{cases} 
\frac{x - x_{\text{max}}}{x_{\text{max}} - x_{\text{min}}} , & x_{\text{min}} \leq x \leq x_{\text{max}} \\
0 , & \text{otherwise} 
\end{cases}
\]

where $S = \bigcup S_i, S_i = \{ x \mid \mu_{\tilde{A}_i}(x) > 0 \}, \tilde{A}_i$ is a triangular fuzzy number, $x_{\text{max}} = \sup S, x_{\text{min}} = \inf S$.

![Figure 3 Membership function of M and G](image)

**Definition 2.4.** If $\tilde{A}_i$ is a triangular fuzzy number, has the right utility value $U_M(\tilde{A}_i)$ based on Maximizing set $M$, and also has the left utility value $U_G(\tilde{A}_i)$ based on Minimizing set $G$, be defined as:

\[
U_M(\tilde{A}_i) = \sup ( \mu_{\tilde{A}_i}(x) \wedge \mu_M(x) )
\]

\[
U_G(\tilde{A}_i) = \sup ( \mu_{\tilde{A}_i}(x) \wedge \mu_G(x) )
\]

Then, the total utility value of fuzzy number $\tilde{A}_i$ is fellowing:

\[
U_T(\tilde{A}_i) = \frac{U_M(\tilde{A}_i) + 1 - U_G(\tilde{A}_i)}{2}
\]

In order to transferring the fuzzy number into letter-grade score, we define the letter-grade universal and the degree of similarity between two fuzzy numbers as fellows.

**Definition 2.5.** Let $L$ be a universal of letter-grade score, $L = \{ A^+, A, A^-, B^+, B, B^- \}$.
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C, F\} where \( A^+, A, A^-, B^+, B, B^-, C \) are represented up to standard grade score and \( F \) is represented down to standard grade score. Then, \( L \) is a letter-grade universal. Furthermore, we define these letter-grade score with the triangular fuzzy number as fellows:

\[
\begin{align*}
A^+ &= (90, 95, 100) \\
A^- &= (85, 90, 95) \\
B^+ &= (80, 85, 90) \\
B^- &= (75, 80, 85) \\
C &= (60, 65, 70) \\
F &= (50, 55, 60)
\end{align*}
\]

Definition 2.6. Degree of similarity between two fuzzy numbers \( X \) and \( Y \): Suppose \( X \) and \( Y \) are two triangular fuzzy numbers. Then the degree of similarity between \( X \) and \( Y \) is denoted by \( S(X, Y) \), and is defined by

\[
S(X, Y) = \frac{U_T(X) \times U_T(Y)}{\max(U_T(X) \times U_T(X), U_T(Y) \times U_T(Y))}
\]

where \( S(X, Y) \in [0, 1] \), \( U_T(X) \) is the utility value of \( X \), and \( U_T(Y) \) is the utility value of \( Y \).

3. Algorithm

In order to evaluating the students’ answerscripts, we use nine satisfication level to evaluate students’ answerscripts in the testing question, i.e., very very good(VVG), superior(SP), very good(VG), good(G), satisfactory to good(SG), satisfactory(S), bad(B), very bad(VB), very very bad(VVB). We use the fuzzy evaluation sheet, be shown in Table 1 to evaluate the students’ answerscripts of \( n \) questions by evaluator. The evaluator evaluates the students’ answerscripys of \( Q_i \) and denote the “√” symbol into fuzzy evaluation grade. Then, we introduce a algorithms for evaluating students’ answerscript as fellows.

<table>
<thead>
<tr>
<th>Question No</th>
<th>Question weight ((Q^W))</th>
<th>Fuzzy evaluation grade</th>
<th>Fuzzy evaluation value ((\tilde{F}EIV))</th>
<th>Fuzzy evaluation score ((\tilde{F}ES))</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Q_1 )</td>
<td></td>
<td>VV G SP VV G S B VB VV</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( Q_2 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( Q_3 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( Q_n )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>1</td>
<td></td>
<td></td>
<td>Fuzzy total score</td>
</tr>
</tbody>
</table>

Letter grade score
Step 1: Transferring the fuzzy evaluation grade of $Q_i$, $i=1,2,...,n$ in the fuzzy evaluation sheet into fuzzy evaluation value ($\tilde{FEV}$) that is a triangular fuzzy number.

**Example.** Suppose the students’ answerscription of $Q_1$ be evaluated as “Good” by evaluator, then transfer the linguistic data “Good” into (76,80,84).

Step 2: Computing the fuzzy evaluation score ($\tilde{FES}_i$) of $Q_i$, $i=1,2,...,n$ is fellowing:

$$\tilde{FES}_i = QW_i \otimes \tilde{FEV}_i, \ i=1,2,...,n$$

where $QW_i \in [0,1]$ and $\sum_{i=1}^{n} QW_i \leq 1$.

**Example.** Suppose the weight of $Q_1$ is 0.15 ($QW_1=0.15$), then $\tilde{FES}_1$ be calculated as fellow:

$$\tilde{FES}_1 = 0.15 \otimes (76, 80, 84)$$
$$= (0.15, 0.15, 0.15) \otimes (76, 80, 84)$$
$$= (11.4, 12.0, 12.6)$$

Step 3: Computing the fuzzy total score is fellowing:

$$\tilde{FTS} = \sum_{i=1}^{n} \tilde{FES}_i$$

Step 4: Transferring the fuzzy total score into letter-grade score.

We calculate the utility of the fuzzy total score and transfer the fuzzy total score into the letter-grade score with the degree of similarity between the fuzzy total score and one of the letter-grade universal.

4. **Example**

We give the student’s answerscriptions evaluation that has 7 questions by evaluator. The evaluation data and the weight of each question are summarized in Table 2.
Evaluating Students’ Answerscripts with Fuzzy Arithmetic

Table 2 Fuzzy evaluation sheet

<table>
<thead>
<tr>
<th>Question No</th>
<th>Question weight ($QW$)</th>
<th>Fuzzy evaluation grade</th>
<th>Fuzzy evaluation value ($FEV$)</th>
<th>Fuzzy evaluation score ($FES$)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>VV G SP VG G SG S B VB VB</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$Q_1$</td>
<td>0.1</td>
<td>√</td>
<td>(87, 90, 93)</td>
<td>(8.7, 9.0, 9.3)</td>
</tr>
<tr>
<td>$Q_2$</td>
<td>0.15</td>
<td>√</td>
<td>(70, 74, 78)</td>
<td>(10.5, 11.1, 11.7)</td>
</tr>
<tr>
<td>$Q_3$</td>
<td>0.25</td>
<td>√</td>
<td>(82, 85, 88)</td>
<td>(20.5, 21.25, 22)</td>
</tr>
<tr>
<td>$Q_4$</td>
<td>0.1</td>
<td>√</td>
<td>(87, 90, 93)</td>
<td>(8.7, 9.0, 9.3)</td>
</tr>
<tr>
<td>$Q_5$</td>
<td>0.1</td>
<td>√</td>
<td>(76, 80, 84)</td>
<td>(7.6, 8.0, 8.4)</td>
</tr>
<tr>
<td>$Q_6$</td>
<td>0.2</td>
<td>√</td>
<td>(76, 80, 84)</td>
<td>(15.2, 16.16, 16.8)</td>
</tr>
<tr>
<td>$Q_7$</td>
<td>0.1</td>
<td>√</td>
<td>(82, 85, 88)</td>
<td>(8.2, 8.5, 8.8)</td>
</tr>
<tr>
<td>Total</td>
<td>1</td>
<td></td>
<td>Fuzzy total score</td>
<td></td>
</tr>
</tbody>
</table>

In the following, we transfer the fuzzy evaluation grade of each question into fuzzy evaluation value. Then, we calculate the fuzzy evaluation score with formula (1) and the calculate fuzzy total score with formula (2). Last, we transfer the fuzzy total score into letter-grade score. These results be summarized in Table 3.

Table 3 Fuzzy evaluation sheet

<table>
<thead>
<tr>
<th>Question No</th>
<th>Question weight ($QW$)</th>
<th>Fuzzy evaluation grade</th>
<th>Fuzzy evaluation value ($FEV$)</th>
<th>Fuzzy evaluation score ($FES$)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>VV G SP VG G SG S B VB VB</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$Q_1$</td>
<td>0.1</td>
<td>√</td>
<td>(87, 90, 93)</td>
<td>(8.7, 9.0, 9.3)</td>
</tr>
<tr>
<td>$Q_2$</td>
<td>0.15</td>
<td>√</td>
<td>(70, 74, 78)</td>
<td>(10.5, 11.1, 11.7)</td>
</tr>
<tr>
<td>$Q_3$</td>
<td>0.25</td>
<td>√</td>
<td>(82, 85, 88)</td>
<td>(20.5, 21.25, 22)</td>
</tr>
<tr>
<td>$Q_4$</td>
<td>0.1</td>
<td>√</td>
<td>(87, 90, 93)</td>
<td>(8.7, 9.0, 9.3)</td>
</tr>
<tr>
<td>$Q_5$</td>
<td>0.1</td>
<td>√</td>
<td>(76, 80, 84)</td>
<td>(7.6, 8.0, 8.4)</td>
</tr>
<tr>
<td>$Q_6$</td>
<td>0.2</td>
<td>√</td>
<td>(76, 80, 84)</td>
<td>(15.2, 16.16, 16.8)</td>
</tr>
<tr>
<td>$Q_7$</td>
<td>0.1</td>
<td>√</td>
<td>(82, 85, 88)</td>
<td>(8.2, 8.5, 8.8)</td>
</tr>
<tr>
<td>Total</td>
<td>1</td>
<td></td>
<td>Fuzzy total score</td>
<td>(79.4, 82.85, 86.3)</td>
</tr>
</tbody>
</table>

Letter grade score $A$

5. Generalized fuzzy evaluation method

In this section, we introduce the generalized fuzzy evaluation method to evaluate the students’ answerscripts with four criteria of each question. We give a generalized fuzzy evaluation sheet, such as Table 4, and assume that the four criteria [14] is following:
c1: Accuracy of information

c2: Adequate

c3: Conciseness

c4: Clear expression

And the weights of every criteria are following:

- Weight of c1 criteria: $CW_1$
- Weight of c2 criteria: $CW_2$
- Weight of c3 criteria: $CW_3$
- Weight of c4 criteria: $CW_4$

Table 4  Generalized fuzzy evaluation sheet

<table>
<thead>
<tr>
<th>Question No</th>
<th>Question weight ($QW$)</th>
<th>criteria weight</th>
<th>Generalized fuzzy evaluation grade</th>
<th>Fuzzy evaluation value ($\tilde{F EV}$)</th>
<th>Fuzzy evaluation score ($\tilde{F ES}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q_1$</td>
<td>$QW_1$</td>
<td>$CW_{11}$</td>
<td>VVG</td>
<td>$\tilde{F EV}_{11}$</td>
<td>$\tilde{F ES}_1$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$CW_{12}$</td>
<td>SP</td>
<td>$\tilde{F EV}_{12}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$CW_{13}$</td>
<td>VG</td>
<td>$\tilde{F EV}_{13}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$CW_{14}$</td>
<td>G</td>
<td>$\tilde{F EV}_{14}$</td>
<td></td>
</tr>
<tr>
<td>$Q_2$</td>
<td>$QW_2$</td>
<td>$CW_{21}$</td>
<td>VG</td>
<td>$\tilde{F EV}_{21}$</td>
<td>$\tilde{F ES}_2$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$CW_{22}$</td>
<td>SP</td>
<td>$\tilde{F EV}_{22}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$CW_{23}$</td>
<td>VG</td>
<td>$\tilde{F EV}_{23}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$CW_{24}$</td>
<td>G</td>
<td>$\tilde{F EV}_{24}$</td>
<td></td>
</tr>
<tr>
<td>$Q_n$</td>
<td>$QW_n$</td>
<td>$CW_{n1}$</td>
<td>VVG</td>
<td>$\tilde{F EV}_{n1}$</td>
<td>$\tilde{F ES}_n$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$CW_{n2}$</td>
<td>SP</td>
<td>$\tilde{F EV}_{n2}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$CW_{n3}$</td>
<td>VG</td>
<td>$\tilde{F EV}_{n3}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$CW_{n4}$</td>
<td>G</td>
<td>$\tilde{F EV}_{n4}$</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>1</td>
<td></td>
<td>Fuzzy total score</td>
<td></td>
<td>Letter-grade score</td>
</tr>
</tbody>
</table>
Furthermore, we introduce a algorithms of the generalized fuzzy evaluation method as fellowing:

Step 1: Transfering the generalized fuzzy evaluation grade of each criteria of $Q_i$, $i=1,2,...,n$ in the generalized fuzzy evaluation sheet into triangular fuzzy number($GG$). Then, calculating the fuzzy evaluation value($FEV$) of each criteria of $Q_i$, $i=1,2,...,n$ is fellowing:

$$FEV_j = CW_j \otimes GG_j, \quad j=1,2,...,4, \quad i=1,2,...,n$$

where $\sum_{i=1}^{n} CW_i = 1$, $0 \leq CW_i \leq 1$.

Step 2: Calculating the fuzzy evaluation score of $Q_i$, $i=1,2,...,n$ is fellowing:

$$FES_i = QW_i \otimes \sum_{j=1}^{4} FEV_j, \quad i=1,2,...,n$$

where $QW_i \in [0,1]$ and $\sum_{i=1}^{n} QW_i = 1$.

Step 3: Calculating the fuzzy total score($FTS$) is fellowing:

$$FTS = \sum_{i=1}^{n} FES_i$$

Step 4: Transfering the fuzzy total score into letter-grade score.

6. Conclusions

In this paper, we introduce a method for evaluating students’ answerscripts. The method transfers the evaluation grade of linguistic data into triangular fuzzy number, and uses the fuzzy arithmetics based on function principle to calculate the fuzzy total score of students’ answerscript. Then, it applies the degree of similarity between fuzzy total score and one of letter-grade universal to transfer fuzzy total score into letter-grade score. Furthermore, we also discuss the generalized fuzzy evaluation with this method.
Reference


