Optimization of Fuzzy Inventory Models under Fuzzy Demand and Fuzzy Lead Time

Chih-Hsun Hsieh

(Received September 9, 2004; Revised October 1, 2004; Accepted October 20, 2004)

Abstract

Inventory model under risk that demand is uncertain is recognized. In this paper, the fuzzy demand per day and fuzzy lead time on a cycle in fuzzy inventory control system are assumed to trapezoidal distribution, trapezoidal fuzzy number by decision maker. A fuzzy inventory model under manager’s preference for order quantity is presented first. This model is given by fuzzy total annual inventory cost summating of total annual holding cost and fuzzy total annual setup cost. We obtain the optimal order quantity by using both Function Principle and Graded Mean Integration Representation method for both computing and representing fuzzy total annual inventory cost. The number of orders in a year, then, is getting by the above optimal order quantity. In addition, we get the reorder point and safety stock under a unit service level by manager. Furthermore, we also introduce a fuzzy inventory model under safety stock based on fuzzy total annual safety stock cost combined by total annual holding cost of safety stock and fuzzy total annual stockout cost. Fuzzy average demand in lead time, mean, thus is calculated by using Graded Mean Integration Representation method. Moreover, we find the optimal reorder point and the optimal safety stock of our second proposed model.

Keywords: Inventory, Fuzzy Inventory; Order Quantity; Function Principle; Graded Mean Integration Representation; Safety Stock; Stockout; Reorder Point; Service Level.

1. Introduction

The classical inventory model (Narasimhan et al., 1995), EOQ, discussed in inventory control system assumed that demand and lead time are constant and known. Once we begin to move closer to reality, we must recognize that demands is never certain but that it occurs with some probability. Inventory models that consider risk attempt to manage the chance of stockout by holding costs, setup costs, and stockout costs. The concept of service level and safety stock is one of methods of processing the inventory problem under stockout risk. The purpose of the safety stock is to cover the random variations in demand and lead time. The safety stock is not intended to cover 100% of the variations during that period. The amount of variation covered by the safety stock depends on the desired stockout risk or the customer service level. When stockout costs are not available, a common surrogate is the customer service level, which

1 Department of Information, General Education, Aletheia University, 32, Chen-Li street, Tamsui 251, Taipei, Taiwan, Republic of China. Corresponding author: e-mail: mailto:jury@email.au.edu.tw.
generally refers to the probability that a demand or a collection of demands are met. In this paper, the unit service level (USL) by manager, one of classified in different ways on service level, is used to decide what should be reorder point and safety stock in both uncertain demand and uncertain lead time? The unit service level can provide us with the exact quantities of units of customer demand filled during each cycle or during any time period, and hence it is a more appropriate measure for many consumer-good applications.

In the real world, fuzzy inventory model with both uncertain parameters and fuzzy variables has been discussed recently. Kacprzyk and Staniewski (1982) present a very interesting approach for aggregate inventory planning. Park (1987) used fuzzy set concept to treat the inventory problem with fuzzy inventory cost under arithmetic operations of Extension Principle. Chen et al. (1996) introduced backorder fuzzy inventory model under Function Principle. Chen and Hsieh (1999) discussed fuzzy inventory models for crisp order quantity or for fuzzy order quantity with generalized trapezoidal fuzzy number. Chang (1999) presented a membership function of the fuzzy total cost of production inventory model, and use Extension principle and centroid method to obtain an estimate of the total cost, and to obtain the economic production quantity. Hsieh (2002) proposed two fuzzy production inventory models for crisp production quantity, or for fuzzy production quantity. This paper finds the optimal solutions of these models by using Extension of the Lagrangean method for solving inequality constrain problem. Hsieh (2003) introduced some fuzzy inventory models under decision maker’s preference. They get the optimal solutions when k=0.5 and all fuzzy parameters are crisp real numbers, can be specified to meet classical inventory model. These models do not discuss the problem of stockout risk which demand per day and lead time are uncertain. Also, we must recognize that demand or lead time is never certain. In this paper, the reorder point under a unit service level by manager in which fuzzy demand per day and fuzzy lead time are assumed to trapezoidal fuzzy number satisfied trapezoidal distribution is introduced. In addition, a fuzzy inventory model under fuzzy annual demand for order quantity based on fuzzy total annual inventory cost first is presented. We derive the optimal order quantity by fuzzy total annual inventory cost with manager’s preference. Furthermore, we also introduce a fuzzy inventory model under safety stock based on fuzzy total annual safety stock cost. The optimal reorder point and the optimal safety stock are solved in our proposed model.

In this paper, we assumed fuzzy demand per day and fuzzy lead time on a cycle to trapezoidal distribution, trapezoidal memberships function. Graded Mean Integration Representation method of Chen and Hsieh (2000) adopted grade as the important degree of each point of support set of generalized fuzzy number. In the fuzzy sense, we know that it is reasonable to discuss the grade of each point of support set of fuzzy number for representing fuzzy number. Therefore, we first discuss the membership function of trapezoidal fuzzy number, and represent the trapezoidal fuzzy number by Graded Mean Integration Representation method for defuzzifying fuzzy total annual inventory cost. Also, in order to calculate fuzzy average
demand in fuzzy lead time, we define the mean of trapezoidal fuzzy number by using Graded Mean Integration Representation method. In addition, we descript some fuzzy arithmetical operations of fuzzy numbers by Chen's Function Principle (1985) for simplifying the calculation, and for keeping the type of membership function of fuzzy number after fuzzy arithmetical operations. In third section, we introduce the reorder point with a unit service level (USL) by manager, which fuzzy demand and fuzzy lead time is satisfying trapezoidal distribution. Furthermore, we derive reorder point by the percentage of stockout level combined by both the area of membership function of fuzzy total demand and the area of membership function of stockout on a cycle. The safety stock also is getting by the above reorder point and the fuzzy average demand. In fourth section, a fuzzy inventory model under fuzzy annual demand for order quantity first is introduced. This model is obtained on fuzzy total annual inventory cost with k-preference by manager combined by both total annual holding cost and fuzzy total annual setup cost. Then, we get the optimal order quantity by using both the operations of Function Principle and Graded Mean Integration Representation method, and obtain the number of orders in a year obtained by the above optimal order quantity and fuzzy annual demand. Moreover, a fuzzy inventory model under safety stock, which demand per day and lead time are uncertain also is presented. The fuzzy demand per day and fuzzy lead time is assumed to trapezoidal fuzzy number satisfied trapezoidal distribution in this model. Our second model is based on fuzzy total annual safety stock cost summating of total annual holding cost of safety stock and fuzzy total annual stockout cost. The optimal reorder point and the optimal safety stock under minimum total annual safety stock cost are found final. In fifth section, we give a numerical example for practicing the following steps derived the optimal solutions in our proposed models. In the following, conclusion and remarks is discussed in final section.

2. Methodology

2.1 Representation and Mean of Trapezoidal Fuzzy Number

More recently, additional important works on the concept of fuzzy number have been written by Dubios and Prade (1980, 1982), and by several other authors. Kaufmann and Gupta (1991) give a definition for fuzzy number that a fuzzy number in R is a fuzzy subset of R that is convex and normal. Thus a fuzzy number can be considered a generalization of the interval of confidence. However, it is not a random variable. A random variable is defined in terms of the theory of probability, which has evolved from theory of measurement. A random variable is an objective datum, whereas a fuzzy number is a subjective datum. It is a valuation, not a measure.

Throughout this paper, we only use popular trapezoidal fuzzy number (TrFN) as the type of all fuzzy parameters in our proposed fuzzy inventory models.
Suppose $\tilde{A}$ is a trapezoidal fuzzy number as shown in Figure 1. It is described as any fuzzy subset of the real line $\mathbb{R}$, whose membership function $\mu_{\tilde{A}}$ satisfies the following conditions.

1. $\mu_{\tilde{A}}(x)$ is a continuous mapping from $\mathbb{R}$ to the closed interval $[0, 1]$,
2. $\mu_{\tilde{A}}(x) = 0$, $-\infty < x \leq a_1$,
3. $\mu_{\tilde{A}}(x) = L(x)$ is strictly increasing on $[a_1, a_2]$,
4. $\mu_{\tilde{A}}(x) = 1$, $a_2 \leq x \leq a_3$,
5. $\mu_{\tilde{A}}(x) = R(x)$ is strictly decreasing on $[a_3, a_4]$,
6. $\mu_{\tilde{A}}(x) = 0$, $a_4 \leq x < \infty$,

where $a_1$, $a_2$, $a_3$, and $a_4$ are real numbers. Also this type of trapezoidal fuzzy number be denoted as $\tilde{A} = (a_1, a_2, a_3, a_4)$.

In addition, Chen and Hsieh introduced Graded Mean Integration Representation method based on the integral value of graded mean h-level of generalized fuzzy number. This method is reasonable that to adopts grade as the important degree of each point of support set of generalized fuzzy number, and to discuss the grade of each point of support set of fuzzy number for representing fuzzy number. Here, we first obtain the representation of trapezoidal fuzzy number $\tilde{A} = (a_1, a_2, a_3, a_4)$, $P_k(\tilde{A})$, by using the method of Chen et al. under $w=1$ and $k$-preference by manager as follow.

$$P_k(\tilde{A}) = \frac{k(a_1 + 2a_2)+(1-k)(2a_3+a_4)}{3}$$

Also, the mean of trapezoidal fuzzy number $\tilde{A} = (a_1, a_2, a_3, a_4)$, $M_{\tilde{A}}$, by using the method of Chen et al. under $w=1$ and $k=0.5$ is defined as follow, and as shown in Figure 1. But we prefer $k=0.5$, since it is does not bias to left or right.

$$M_{\tilde{A}} = \frac{a_1 + 2a_2 + 2a_3 + a_4}{6}$$

For example, suppose $\tilde{A} = (1, 2, 3, 7)$ is a trapezoidal fuzzy number, then the representation of $\tilde{A}$ and the mean of $\tilde{A}$ can be calculated by Formula (1) and (2) respectively, as follows.

$$P_k(\tilde{A}) = \frac{5k + 13(1-k)}{3} = \frac{13 - 8k}{3}$$
$$M_{\tilde{A}} = \frac{1 + 4 + 6 + 7}{6} = 3$$

2.2 The Fuzzy Arithmetical Operations under Function Principle

In this paper, we use the Function Principle to simplify the calculation. Function Principle in fuzzy theory is used as the computational model avoiding the complications which can be
caused by the operations using Extension Principle. We describe some fuzzy arithmetical operations under Function Principle as follows.

Suppose $\tilde{A} = (a_1, a_2, a_3, a_4)$ and $\tilde{B} = (b_1, b_2, b_3, b_4)$ are two trapezoidal fuzzy numbers. Then,

1. **The addition of $\tilde{A}$ and $\tilde{B}$ is**
   
   \[ \tilde{A} \oplus \tilde{B} = (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4), \]
   
   where $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4$ are any real numbers.

2. **The multiplication of $\tilde{A}$ and $\tilde{B}$ is**
   
   \[ \tilde{A} \otimes \tilde{B} = (c_1, c_2, c_3, c_4), \]
   
   where $T = \{ a_1b_1, a_1b_4, a_4b_1, a_4b_4 \}$, $T_1 = \{ a_2b_2, a_2b_3, a_3b_2, a_3b_3 \}$, $c_1 = \min T$, $c_2 = \min T_1$, $c_3 = \max T_1$, $c_4 = \max T$.
   
   Also, if $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4$ are all nonzero positive real numbers, then
   
   \[ \tilde{A} \otimes \tilde{B} = (a_1b_1, a_2b_2, a_3b_3, a_4b_4), \]
   
   where $\tilde{A} \otimes \tilde{B}$ is a trapezoidal fuzzy number.

3. **The subtraction of $\tilde{A}$ and $\tilde{B}$ is**
   
   \[ \tilde{A} \Theta \tilde{B} = (a_1 - b_4, a_2 - b_3, a_3 - b_2, a_4 - b_1), \]
   
   where $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4$ are any real numbers.

4. **The division of $\tilde{A}$ and $\tilde{B}$ is**
   
   \[ \tilde{A} \oslash \tilde{B} = (a_1/b_4, a_2/b_3, a_3/b_2, a_4/b_1). \]

5. **Let $\alpha \in R$, then**
   
   \[
   \begin{cases}
   (i) \alpha \geq 0, \alpha \otimes \tilde{A} = (\alpha a_1, \alpha a_2, \alpha a_3, \alpha a_4), \\
   (ii) \alpha < 0, \alpha \otimes \tilde{A} = (\alpha a_4, \alpha a_3, \alpha a_2, \alpha a_1).
   \end{cases}
   \]

For example, suppose $\tilde{A} = (1, 2, 3, 4)$, $\tilde{B} = (1, 3, 4, 6)$ are two trapezoidal fuzzy numbers, and $\alpha = 2.5$ then

1. $\tilde{A} \oplus \tilde{B} = (2, 5, 7, 10)$,
2. $\tilde{A} \otimes \tilde{B} = (1, 6, 12, 24)$,
3. $\alpha \otimes \tilde{B} = (2.5, 7.5, 10, 15)$,
4. $\tilde{B} \oslash \alpha = \tilde{B} \otimes (\alpha^{-1}) = (0.4, 1.2, 1.6, 2.4)$.

3. **Reorder Point and Safety Stock under a Unit Service Level**
Narasimhan et al. (1995) discussed unit service level that indicates the percentage of units of demand filled during any period of time, whereas the unit stockout risk (USOR) specifies the quantities of units unfilled or short during that period. For example, suppose that the inventory manager is willing to accept a 95% of unit service level, \( S_p = 0.95 \), on any cycle. On the other word, we have the percent of stockout level on a cycle, such as \( 1 - S_p = 0.05 \). In this paper, we suppose fuzzy demand per day \( \tilde{D}_p = (dp_1, dp_2, dp_3, dp_4) \) and fuzzy lead time \( \tilde{L} = (l_1, l_2, l_3, l_4) \) that be decided by inventory manager, as TrFN distribution approached. Therefore, the fuzzy total demand in fuzzy lead time \( \tilde{D}_L \) is equal to fuzzy demand per day multiplied by fuzzy lead time by using Function Principle, such as \( \tilde{D}_L = \tilde{D}_p \otimes \tilde{L} = (d_1, d_2, d_3, d_4) \), and is a trapezoidal fuzzy number as shown in Figure 2.

![Figure 1. The TrFN Distribution of Trapezoidal Fuzzy Number \( \tilde{\lambda} = (a_1, a_2, a_3, a_4) \) and the Mean of \( \tilde{\lambda}, M_{\tilde{\lambda}} \)](image)

In Figure 2, the symbol \( R \) is represented to reorder point on a cycle, \( M_{DL} \) is the mean of fuzzy total demand in fuzzy lead time, and the area of triangular \( TRd_4 \) is percent of stockout. In addition, the safety stock \( (S_s) \) can be determined using \( R - M_{DL} \) on a cycle. What, then, should be the reorder point satisfying the percentage of unit service level on a cycle? Here, we suppose that the inventory manager indicated that she really had meant a \( S_p \) percentage of unit service level. Then the percentage of stockout be calculated as \( 1 - S_p \), we have

\[
1 - S_p = \frac{\text{the area of triangular TRd}_4}{\text{the area of } \tilde{D}_L},
\]

where the area of triangular \( TRd_4 = (d_4 - R)^2 / 2(d_4 - d_3) \), and

the area of \( \tilde{D}_L = (d_2 - d_1) / 2 + (d_3 - d_2) + (d_4 - d_3) / 2 \).

From Formula (3), we get the reorder point, \( R \), as

\[
R = d_4 - \sqrt{(d_4 - d_1 - d_2 + d_3)(d_4 - d_3)(1 - S_p)}.
\]

Also, we find the safety stock, \( S_s \), as
Optimization of Fuzzy Inventory Models under Fuzzy Demand and Fuzzy Lead Time

\[ S_s = R - M_{\tilde{D}_L}, \]  

(5)

where \( M_{\tilde{D}_L} = (d_1 + 2d_2 + 2d_3 + d_4) / 6 \) by Formula (2).

![Figure 2. The TrFN Distribution of Fuzzy Total Demand in Fuzzy Lead Time (\( \tilde{D}_L \))](image)

For example, suppose that inventory manager really had meant a 95% unit service level, fuzzy demand per day \( \tilde{D}_p = (8, 9, 11, 12) \), and fuzzy lead time \( L = (3, 4, 6, 7) \). The following steps are necessary for calculating the safety stock (\( S_s \)) under a 95% unit service level:

1. Specify unit service level; \( S_p = 0.95 \).
2. Calculate the percent of stockout level; \( 1 - S_p = 0.05 \).
3. Calculate the fuzzy total demand in fuzzy lead time \( \tilde{D}_L \); \( \tilde{D}_L = \tilde{D}_p \otimes L = (8, 9, 11, 12) \otimes (3, 4, 6, 7) = (24, 36, 66, 84) \).
4. Get the reorder point (\( R \)) by Formula (4); \( R = 75 \).
5. Calculate the mean of fuzzy total demand in fuzzy lead time, \( M_{\tilde{D}_L} \), by Formula (2); \( M_{\tilde{D}_L} = 52 \).
6. Find the safety stock (\( S_s \)) by Formula (5); \( S_s = 75 - 52 = 23 \).

Using a comport program we obtain results with the fuzzy demand per day \( \tilde{D}_p \) and the fuzzy lead time \( L \) in the above example under some unit service levels. The results are summarized in Table 1.

4. Optimization of Fuzzy Inventory Models under Fuzzy Demand and Fuzzy Lead Time

Throughout this paper, we assume the following variables in order to simplify the treatment of our proposed fuzzy inventory models:

\( \tilde{D} \): fuzzy annual demand,

\( C_c \): holding cost per unit,
Chih-Hsun Hsieh

\[ C_o: \text{setup cost per order}, \]
\[ Q: \text{order quantity per order}, \]
\[ C_s: \text{stockout cost per unit}, \]
\[ \ddot{D}_p: \text{fuzzy demand per day}, \]
\[ \ddot{L}: \text{fuzzy lead time on a cycle}, \]
\[ \dddot{N}: \text{number of orders per year}; \]
\[ \dddot{S}: \text{expected stockout per order}; \]
\[ \dddot{D}_L: \text{fuzzy total demand in fuzzy lead time}; \]
\[ M_{DL}: \text{mean of } \dddot{D}_L; \]
\[ N: \text{average number of orders per year}; \]
\[ S_p: \text{unit service level by manager}, \]
\[ R: \text{reorder point on a cycle}, \]
\[ S_c: \text{safety stock on a cycle}. \]

Table 1. The Results by Computer Program

<table>
<thead>
<tr>
<th>Service level</th>
<th>Stockout level</th>
<th>Reorder point</th>
<th>Safety stock</th>
<th>Service level</th>
<th>Stockout level</th>
<th>Reorder point</th>
<th>Safety stock</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.81</td>
<td>0.19</td>
<td>66.46</td>
<td>14.46</td>
<td>0.91</td>
<td>0.09</td>
<td>71.93</td>
<td>19.93</td>
</tr>
<tr>
<td>0.82</td>
<td>0.18</td>
<td>66.92</td>
<td>14.92</td>
<td>0.92</td>
<td>0.08</td>
<td>72.62</td>
<td>20.62</td>
</tr>
<tr>
<td>0.83</td>
<td>0.17</td>
<td>67.40</td>
<td>15.40</td>
<td>0.93</td>
<td>0.07</td>
<td>73.35</td>
<td>21.35</td>
</tr>
<tr>
<td>0.84</td>
<td>0.16</td>
<td>67.90</td>
<td>15.90</td>
<td>0.94</td>
<td>0.06</td>
<td>74.14</td>
<td>22.14</td>
</tr>
<tr>
<td>0.85</td>
<td>0.15</td>
<td>68.41</td>
<td>16.41</td>
<td>0.95</td>
<td>0.05</td>
<td>75.00</td>
<td>23.00</td>
</tr>
<tr>
<td>0.86</td>
<td>0.14</td>
<td>68.94</td>
<td>16.94</td>
<td>0.96</td>
<td>0.04</td>
<td>75.95</td>
<td>23.95</td>
</tr>
<tr>
<td>0.87</td>
<td>0.13</td>
<td>69.49</td>
<td>17.49</td>
<td>0.97</td>
<td>0.03</td>
<td>77.03</td>
<td>25.03</td>
</tr>
<tr>
<td>0.88</td>
<td>0.12</td>
<td>70.06</td>
<td>18.06</td>
<td>0.98</td>
<td>0.02</td>
<td>78.31</td>
<td>26.31</td>
</tr>
<tr>
<td>0.89</td>
<td>0.11</td>
<td>70.65</td>
<td>18.65</td>
<td>0.99</td>
<td>0.01</td>
<td>79.98</td>
<td>27.98</td>
</tr>
<tr>
<td>0.90</td>
<td>0.10</td>
<td>71.27</td>
<td>19.27</td>
<td>1.00</td>
<td>0.00</td>
<td>83.98</td>
<td>31.98</td>
</tr>
</tbody>
</table>

* The mean of fuzzy total demand on a cycle= 52.

4.1 Optimization of Fuzzy Inventory Model for Order Quantity

Now, we first introduce the fuzzy inventory model with manager’s k-preference for order quantity Q. This model is based on fuzzy total annual inventory cost, FTC, summating of total annual holding cost and fuzzy total annual order cost. Then, according to EOQ inventory system, the total annual holding cost (THC) and the fuzzy total annual setup cost (FTSC) are defined, respectively, as follows.

\[ THC = C_c \otimes \frac{Q}{2}, \]

and

\[ FTSC = C_o \otimes (\dddot{D} \otimes Q). \]
Thus, the fuzzy total annual inventory cost in this model is given by

\[ FTC = THC + FTSC = C_c \times \frac{Q}{2} \otimes C_o \otimes (\tilde{D} \otimes Q) \]  

(6)

where \( \otimes \), \( \otimes \), and \( \oplus \) are the fuzzy arithmetical operations by Function Principle.

In Formula (6), \( Q/2 \) represents average inventory depended heavily on the assumption of a constant sales rate producing a linear decline in the inventory position, and \( \tilde{D} \otimes Q \) is the number of orders.

In the following, we solve the optimal order quantity of this model by Formula (6). First the fuzzy annual demand is assumed to trapezoidal fuzzy number as \( \tilde{D} = (D_1, D_2, D_3, D_4) \). Then we get the fuzzy total annual inventory cost (FTC) by Formula (6) as

\[ FTC = \frac{C_Q}{2} + \frac{C_D_1}{Q}, \frac{C_Q}{2} + \frac{C_D_2}{Q}, \frac{C_Q}{2} + \frac{C_D_3}{Q}, \frac{C_Q}{2} + \frac{C_D_4}{Q}. \]

Also, the fuzzy total annual inventory cost is represented with manager’s k-preference by using Formula (1). The result is

\[ P_k(FTC) = \frac{1}{3} \left[ \frac{3C_Q}{2} + \frac{kC_o(D_1 + 2D_2) + (1-k)C_o(2D_3 + D_4)}{Q} \right]. \]

Then, we can get the optimal order quantity \( Q^* \) when \( P_k(FTC) \) is minimization. In order to find the minimization of \( P_k(FTC) \), the derivative of \( P_k(FTC) \) with \( Q \) is

\[ \frac{\partial P_k(FTC)}{\partial Q} = \frac{1}{3} \left[ \frac{3C_Q}{2} - \frac{kC_o(D_1 + 2D_2) + (1-k)C_o(2D_3 + D_4)}{Q^*} \right]. \]

Let \( \frac{\partial P_k(FTC)}{\partial Q} = 0 \), we find the optimal order quantity \( Q^* \) as

\[ Q^* = \frac{2C_o \cdot k(D_1 + 2D_2) + (1-k)(2D_3 + D_4)}{3C_c}. \]

(7)

In addition, the number of orders on a year, \( \tilde{N} \), also is given by

\[ \tilde{N} = \tilde{D} \otimes Q^* = (\frac{D_1}{Q}, \frac{D_2}{Q}, \frac{D_3}{Q}, \frac{D_4}{Q}). \]

(8)

Then, the average number of orders per year (N), that is the mean of \( \tilde{N} \), is obtained by Formula (2) and (8). The result is
4.2 Optimization of Fuzzy Inventory Model under Safety Stock

Furthermore, we also present a fuzzy inventory model under safety stock when demands per day and lead time are uncertain. This model is given by fuzzy total annual safety stock cost summating of total annual holding cost of safety stock and fuzzy total annual stockout cost. Then, the fuzzy total annual safety stock cost, FTSSC, is obtained by

\[
FTSSC = C_c \times S_s \oplus C_s \otimes (\bar{D} \otimes Q^*) \otimes \bar{s}
\]  

(10)

where \(\otimes, \oplus\), and \(\otimes\) are the fuzzy arithmetical operations by Function Principle.

In Formula (10), the safety stock \(S_s\) can also be represented by \(R - M_{\bar{D}_L}\) by Formula (5) with fuzzy total demand in fuzzy lead time, \(\bar{D}_L = (d_1, d_2, d_3, d_4)\), as shown in Figure 2. In addition, \((\bar{D} \otimes Q^*)\) represents the number of orders per year, \((\bar{N}, \bar{N} = \bar{D} \otimes Q^*)\), and the result is a trapezoidal fuzzy number after calculating. Also, the expected stockout per order \((\bar{s})\) is combined by the average stockout per orders and the percentage of the average stockout. According to the assumption of a constant demand rate producing a linear decline in the inventory position. For such a linear function, declined between maximum stockout and minimum stockout, the average falls at the midpoint or geometric balance point of \((\text{maximum stockout} + \text{minimum stockout})/2\). The average stockout per orders then is given by \((d_4 - R)/2\) where the maximum stockout is \(d_4 - R\) in which the maximum demand is \(d_4\) in fuzzy total demand in fuzzy lead time \((\bar{D}_L)\) and the reorder point is \(R\), and the minimum stockout is zero. Furthermore, the percentage of the above average stockout is obtained by the same method such as calculating the percentage of stockout \((1 - S_p)\), Formula (3), as follow.

\[
\frac{(d_4 - R)^2}{2(d_4 - d_1)(d_4 + d_1 - d_2 - d_3)}.
\]

Thus, by Formula (10) the fuzzy total annual safety stock cost also is given by

\[
FTSSC = C_c \times (R - M_{\bar{D}_L}) \oplus C_s \otimes \bar{N} \otimes \frac{(d_4 - R)^2}{2(d_4 - d_1)(d_4 + d_1 - d_2 - d_3)}.
\]  

(11)

In the following, by Formula (11) we solve the optimal reorder point \(R^*\) of this model with the above optimal order quantity \(Q^*\) of our first proposed model in Formula (7). The fuzzy annual demand and the fuzzy total demand in fuzzy lead time is assumed to trapezoidal fuzzy number as \(\bar{D} = (D_1, D_2, D_3, D_4)\) and \(\bar{D}_L = (d_1, d_2, d_3, d_4)\), respectively. We first use the average number of orders per year \((\bar{N})\) by Formula (9) to approach \(\bar{N}\) in Formula (11) for defuzzifying the number of orders per year \((\bar{N})\), then the fuzzy total annual safety stock cost becomes
Optimization of Fuzzy Inventory Models under Fuzzy Demand and Fuzzy Lead Time

\[ \text{FTSSC} = C_c \times (R - M_{DL}) + C_s \times N \times \frac{(d_u - R)^2}{2(d_u - d_l)(d_u + d_l - d_i - d_j)} \]

Also, the above fuzzy total annual safety stock can be simplified as

\[ \text{FTSSC} = C_c R - M_{DL} \times C_c + \frac{3}{4} \times \frac{N}{t} \times d_u - 3t \times d_u^2 R + 3t \times d_u^2 R - tR^3 \]

where set \( t = \frac{C_c N}{4(d_u - d_l)(d_u + d_l - d_i - d_j)} \), and \( t \) is a constant.

Here, we can get the optimal reorder point \( R^* \) in Formula (12) when \( \text{FTSSC} \) is minimization. In order to find minimization \( \text{FTSSC} \), the derivative of \( \text{FTSSC} \) with \( R \) is

\[ \frac{d\text{FTSSC}}{dR} = C_c - 3t \times d_u^2 + 6t \times d_u R - 3tR^2 \]

Let \( \frac{d\text{FTSSC}}{dR} = 0 \), we find the optimal reorder point \( R^* \) as

\[ R^* = d_u - \sqrt[3]{\frac{C_c}{3t}} \]  \hspace{1cm} (13)

Then the optimal safety stock \( S_s^* \) is

\[ S_s^* = R^* - M_{DL} \]  \hspace{1cm} (14)

5. Numerical Example

The Brown computer company’s manager estimates computer mouse annual demand about 10,000 units, and she gives a 0.5 preference of trapezoidal membership function about fuzzy total annual inventory cost. The setup cost is $150 per order, and the holding cost is $0.75 per unit per year. In addition, the uncertain lead time is about 10 days, and the stockout cost per unit is $2. What safety stock should be kept for obtaining the minimization of fuzzy total annual safety stock cost?

First, we use the following steps to derive the optimal order quantity (\( Q^* \)) by using our first proposed model, Formula (6).

1. Specify the setup cost per order and the holding cost, \( C_o = 150 \) and \( C_c = 0.75 \), respectively.
2. Assume the trapezoidal fuzzy number \( \tilde{D} = (9400, 9900, 10100, 10600) \) to approach the uncertain annual demand (about 10,000 units).
3. Specify the manager’s preference on the fuzzy total annual inventory cost (FTC); \( k = 0.5 \).
4. Get the optimal order quantity with manager’s k-preference by Formula (7); \( Q^* = 2000 \).
5. Find the average number of orders per year (N) by Formula (9); N = 5.

Second, the following steps are necessary for calculating the optimal reorder point (R*) and the optimal safety stock (Ss*) by using the other our proposed model, Formula (10).
1. Specify the stockout cost; C_s = 2.
2. Assume the trapezoidal fuzzy number \( \tilde{\tau} = (8, 9, 11, 12) \) to approach the fuzzy lead time on a cycle (about 10 days).
3. Assume company has 300 work days on a year, and calculate the fuzzy total demand in fuzzy lead time \( \tilde{D}_L = (d_1, d_2, d_3, d_4); \quad \tilde{D}_L = (\tilde{\tau} \otimes 300) \otimes \tilde{\tau} = (251, 297, 370, 424). \)
4. Calculate the t constant in Formula (12); \( t = 0.0002. \)
5. Obtain the optimal reorder point (R*); R* = 388.
6. Calculate the mean of fuzzy total demand in fuzzy lead time, \( M_{DL} \), by Formula (2); \( M_{DL} = 335. \)
7. Find the optimal safety stock (Ss*) by Formula (14); Ss* = 53.
8. Calculate the minimization of fuzzy total annual safety stock cost by Formula (12); \( FTSSC = 49. \)

In addition, here applied to the above example with some different k-preference values of manager by using a computer program, the results then can be supporting for decision maker to decide what optimal order quantity should be need under different k-preferences, and what optimal safety stock should be kept. These results are summarized in Table 2. Here we can see that the each optimal result under a different k-preference in Formula (10) is all the same value, that is \( FTSSC = 49, R_* = 388 \) and \( Ss* = 53 \), since the each average number of orders per year (N) by Formula (9) in the above example just almost approaches to the same value, N=5.

6. Remarks and Conclusion

In this paper, Function Principle is used to simplify the calculation of fuzzy total annual inventory cost, or fuzzy total demand in fuzzy lead time. We do not introduce a new addition symbol, as the sum under the Extension Principle is the same as shown in Figure 3. For a mathematically minded reader, Chen (1985) observe that the Extension Principle is a form of convolution while the Function Principle is akin to a pointwise multiplication. If instead we were to use the Extension Principle, it could be seen that multiplication of two trapezoidal fuzzy numbers would result in a drum-shape, as shown in Figure 4. Thus the calculation will be further complicated if we decide to use Extension Principle to find the optimal solutions in our proposed models.

In addition, in the fuzzy environment, our first proposed model, Formula (6), may be possible and reasonable to discuss the fuzzy inventory model for order quantity Q with
preference of fuzzy total annual inventory cost by decision maker. Also we find that the optimal order quantity $Q^*$ will becomes

$$Q^* = \sqrt{\frac{2C_D}{C_i}}$$

when fuzzy annual demand is a crisp real number ($D$), that is $\bar{D} = (D_1, D_2, D_3, D_4) = (D, D, D, D) = D$, and decision maker prefer the mean of fuzzy total annual inventory cost, $k=0.5$. It means that the optimal solution of our proposed model can be specified to meet the classical inventory model. Hence this fuzzy inventory model is executable and useful in the real world.

Extensive numerical simulations, Wang (1997) compared the effect of defuzzification on the inputs to defuzzification on the output. The results were numerically close, but with one significant feature. Defuzzification before the calculation leads to a slightly lower values. We leave the analysis of this phenomenon for further research. Also, our further research will discuss uncertain datum, such as fuzzy demand and fuzzy lead time, how to represent to fuzzy number more generally.

### Table 2. Example Results

<table>
<thead>
<tr>
<th>k value</th>
<th>Fuzzy inventory model for order quantity, Formula(6)</th>
<th>Fuzzy inventory model for safety stock, Formula(10)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Minimum fuzzy total annual inventory cost ($FTC^*_c$)</td>
<td>Optimal order quantity $Q^*$</td>
</tr>
<tr>
<td>k=0</td>
<td>1520</td>
<td>2026</td>
</tr>
<tr>
<td>k=0.1</td>
<td>1516</td>
<td>2021</td>
</tr>
<tr>
<td>k=0.2</td>
<td>1512</td>
<td>2016</td>
</tr>
<tr>
<td>k=0.3</td>
<td>1508</td>
<td>2011</td>
</tr>
<tr>
<td>k=0.4</td>
<td>1504</td>
<td>2005</td>
</tr>
<tr>
<td>k=0.5</td>
<td>1500</td>
<td>2000</td>
</tr>
<tr>
<td>k=0.6</td>
<td>1496</td>
<td>1995</td>
</tr>
<tr>
<td>k=0.7</td>
<td>1492</td>
<td>1989</td>
</tr>
<tr>
<td>k=0.8</td>
<td>1488</td>
<td>1984</td>
</tr>
<tr>
<td>k=0.9</td>
<td>1484</td>
<td>1979</td>
</tr>
<tr>
<td>k=1</td>
<td>1480</td>
<td>1973</td>
</tr>
</tbody>
</table>
Figure 3. The Fuzzy Addition Operation of Function Principle and Extension Principle

Figure 4. The Comparing of Fuzzy Multiplication Operation under Function Principle and Extension Principle

References


